

# TRIANGULATED ENDOFUNCTORS OF THE DERIVED CATEGORY OF COHERENT SHEAVES WHICH DO NOT ADMIT DG LIFTINGS

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**ABSTRACT.** In [RV], Rizzardo and Van den Bergh constructed an example of a triangulated functor between the derived categories of coherent sheaves on smooth projective varieties over a field  $k$  of characteristic 0 which is not of the Fourier-Mukai type. The purpose of this note is to show that if  $\text{char } k = p$  then there are very simple examples of such functors. Namely, for a smooth projective  $Y$  over  $\mathbb{Z}_p$  with the special fiber  $i : X \hookrightarrow Y$ , we consider the functor  $Li^* \circ i_* : D^b(X) \rightarrow D^b(X)$  from the derived categories of coherent sheaves on  $X$  to itself. We show that if  $Y$  is a flag variety which is not isomorphic to  $\mathbb{P}^1$  then  $Li^* \circ i_*$  is not of the Fourier-Mukai type. Note that by a theorem of Toen ([T], Theorem 8.15) the latter assertion is equivalent to saying that  $Li^* \circ i_*$  does not admit a lifting to a  $\mathbb{F}_p$ -linear DG quasi-functor  $D_{dg}^b(X) \rightarrow D_{dg}^b(X)$ , where  $D_{dg}^b(X)$  is a (unique) DG enhancement of  $D^b(X)$ . However, essentially by definition,  $Li^* \circ i_*$  lifts to a  $\mathbb{Z}_p$ -linear DG quasi-functor.

Given smooth proper schemes  $X_1, X_2$  over a field  $k$  and an object  $E \in D^b(X_1 \times X_2)$  of the bounded derived category of coherent sheaves on  $X_1 \times X_2$  define a triangulated functor

$$(0.1) \quad \Phi_E : D^b(X_1) \rightarrow D^b(X_2)$$

sending a bounded complex  $M$  of coherent sheaves on  $X_1$  to  $Rp_{2*}(E \otimes^L p_1^* M)$ , where  $p_i : X_1 \times X_2 \rightarrow X_i$  are the projections. Recall that a triangulated functor  $D^b(X_1) \rightarrow D^b(X_2)$  is said to be of the Fourier-Mukai type if it is isomorphic to  $\Phi_E$  for some  $E$ .

Let  $Y$  be a smooth projective scheme over  $\text{Spec } \mathbb{Z}_p$  and let  $X$  be its special fiber,  $i : X \hookrightarrow Y$  the closed embedding. Consider the triangulated functor  $G : D^b(X) \rightarrow D^b(X)$  given by the formula

$$G = Li^* \circ i_*$$

We shall see that in general  $G$  is not of the Fourier-Mukai type.

**Theorem 1.** *Let  $Z$  a smooth projective scheme over  $\text{Spec } \mathbb{Z}_p$ ,  $Y = Z \times Z$ ,  $X = Y \times_{\text{Spec } \mathbb{Z}_p} \text{Spec } \mathbb{F}_p$ . Assume that*

- (1) *The Frobenius morphism  $Fr : \overline{Z} \rightarrow \overline{Z}$ , where  $\overline{Z} = Z \times \text{Spec } \mathbb{F}_p$ , does not lift modulo  $p^2$ .*
- (2)  *$H^1(X, T_X) = 0$ , where  $T_X$  is the tangent sheaf on  $X$ .*

*Then  $G = Li^* \circ i_* : D^b(X) \rightarrow D^b(X)$  is not of the Fourier-Mukai type.*

For example, let  $GL_n$  be the general linear group over  $\text{Spec } \mathbb{Z}_p$ ,  $B \subset GL_n$  a Borel subgroup. Then, by Theorem 6 from [BTLM], for any  $n > 2$ , the flag variety  $Z = GL_n/B$  satisfies the first assumption of the Theorem *i.e.*, the Frobenius  $Fr : \overline{Z} \rightarrow \overline{Z}$  does not lift on  $Z \times \text{Spec } \mathbb{Z}/p^2\mathbb{Z}$ . By ([KLT], Theorem 2), we have that  $H^1(\overline{Z}, T_{\overline{Z}}) =$

$H^1(\overline{Z}, \mathcal{O}_{\overline{Z}}) = 0$ . It follows that  $H^1(X, T_X) = 0$ . Hence, by the Theorem, for  $n > 2$ ,  $G : D^b(X) \rightarrow D^b(X)$  is not of the Fourier-Mukai type.

*Proof.* Assume the contrary and let  $E \in D^b(X \times X)$  be the Fourier-Mukai kernel. By definition, for every  $M \in D^b(X)$  we have a functorial isomorphism

$$(0.2) \quad G(M) \xrightarrow{\sim} R p_{2*}(E \overset{L}{\otimes} p_1^* M).$$

By the projection formula ([H], Chapter II, Prop. 5.6) we have that

$$i_* \circ Li^* \circ i_*(M) \xrightarrow{\sim} i_*(M) \overset{L}{\otimes} i_*(\mathcal{O}_X) \xrightarrow{\sim} i_*(M) \otimes (\mathcal{O}_Y \xrightarrow{p} \mathcal{O}_Y) \xrightarrow{\sim} i_*(M) \oplus i_*(M)[1]$$

In particular, if  $M$  is a coherent sheaf then  $\underline{H}^i(G(M)) \simeq M$  for  $i = 0, -1$  and  $\underline{H}^i(G(M)) = 0$  otherwise. Applying this observation and formula (0.2) to skyscraper sheaves,  $M = \delta_x$ ,  $x \in X(\overline{\mathbb{F}}_p)$ , we conclude that the coherent sheaves  $\underline{H}^i(E)$  are set theoretically supported on the diagonal  $\Delta_X \subset X \times X$ . Applying the same formulas to  $M = \mathcal{O}_X$  we see that  $p_{2*}(\underline{H}^i(E)) = \mathcal{O}_X$  for  $i = 0, -1$  and  $p_{2*}(\underline{H}^i(E)) = 0$  otherwise. In fact, every coherent sheaf  $F$  on  $X \times X$  which is set theoretically supported on the diagonal and such that  $p_{2*}F = \mathcal{O}_X$  is isomorphic to  $\mathcal{O}_{\Delta_X}$ . It follows that  $\underline{H}^0(E) = \underline{H}^{-1}(E) = \mathcal{O}_{\Delta_X}$ . In the other words,  $E$  fits into an exact triangle in  $D^b(X \times X)$

$$(0.3) \quad \mathcal{O}_{\Delta_X}[1] \xrightarrow{\alpha} E \longrightarrow \mathcal{O}_{\Delta_X} \xrightarrow{\beta} \mathcal{O}_{\Delta_X}[2]$$

for some  $\beta \in Ext_{\mathcal{O}_{X \times X}}^2(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X})$ . We wish to show that the second assumption in the Theorem implies that  $\beta = 0$ , while the first one implies that  $\beta \neq 0$ . For every  $M \in D^b(X)$ , (0.3) gives rise to an exact triangle

$$(0.4) \quad M[1] \xrightarrow{\alpha_M} G(M) \longrightarrow M \xrightarrow{\beta_M} M[2]$$

Our main tool is the following result.

**Lemma 0.1.** *For a coherent sheaf  $M$  the following conditions are equivalent.*

- (1)  $\beta_M = 0$ .
- (2)  $G(M) \xrightarrow{\sim} M \oplus M[1]$ .
- (3) *There exists a morphism  $\lambda : G(M) \rightarrow M[1]$  such that  $\lambda \circ \alpha_M$  is an isomorphism.*
- (4)  *$M$  admits a lift modulo  $p^2$  i.e., there is a coherent sheaf  $\tilde{M}$  on  $Y$  flat over  $\mathbb{Z}/p^2\mathbb{Z}$  such that  $i^*\tilde{M} \simeq M$ .*

*Proof.* The equivalence of (1), (2) and (3) is immediate. Let us check that (3) is equivalent to (4). A morphism  $\lambda : G(M) \rightarrow M[1]$  gives rise by adjunction a morphism  $\gamma : i_*M \rightarrow i_*M[1]$ . Note that  $\tilde{M} = \text{cone } \gamma[1]$  is a coherent sheaf on  $Y$  which is an extension of  $i_*M$  by itself. It suffices to prove that  $\lambda \circ \alpha_M : M[1] \rightarrow M[1]$  is an isomorphism if and only if  $\tilde{M}$  is flat over  $\mathbb{Z}/p^2\mathbb{Z}$ . Indeed, from the exact triangle

$$Li^*i_*M \rightarrow Li^*(\tilde{M}) \rightarrow Li^*i_*M \rightarrow Li^*i_*M[1]$$

we get a long exact sequence of the cohomology sheaves

$$0 \rightarrow M \rightarrow L_1i^*(\tilde{M}) \rightarrow M \xrightarrow{\lambda \circ \alpha_M} M \rightarrow i^*(\tilde{M}) \rightarrow M \rightarrow 0$$

Thus  $\lambda \circ \alpha_M$  is an isomorphism if and only if in the exact sequence

$$0 \rightarrow i_*M \rightarrow \tilde{M} \rightarrow i_*M \rightarrow 0$$

the image of second map is the kernel of the multiplication by  $p$  on  $\tilde{M}$  and also the image of this map. The latter is equivalent to flatness of  $\tilde{M}$  over  $\mathbb{Z}/p^2\mathbb{Z}$ .  $\square$

We have a spectral sequence converging to  $Ext_{\mathcal{O}_{X \times X}}^*(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X})$  whose second page is  $H^*(X, \mathcal{E}xt_{\mathcal{O}_{X \times X}}^*(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}))$ . In particular, we have a homomorphism

$$Ext_{\mathcal{O}_{X \times X}}^2(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}) \rightarrow H^0(X, \mathcal{E}xt_{\mathcal{O}_{X \times X}}^2(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X})) \xrightarrow{\sim} H^0(X, \wedge^2 T_X).$$

Let us check that the image  $\mu$  of  $\beta$  under this map is 0. To do this we apply the Lemma to skyscraper sheaves  $\delta_x$ , where  $x$  runs over closed points of  $X$ . On the one hand, the evaluation of the bivector field  $\mu$  at  $x$  is equal to the class of  $\beta_{\delta_x}$  in  $Ext_{\mathcal{O}_X}^2(\delta_x, \delta_x) \xrightarrow{\sim} \wedge^2 T_{x,X}$ . On the other hand, by the Lemma,  $\beta_{\delta_x} = 0$  since  $\delta_x$  is liftable modulo  $p^2$ . Next, the assumption that  $H^1(X, T_X) = 0$  implies that  $\beta$  lies in the image of the map

$$(0.5) \quad v : H^2(X, \mathcal{O}_X) \xrightarrow{\sim} H^2(X, \mathcal{E}xt_{\mathcal{O}_{X \times X}}^0(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X})) \rightarrow Ext_{\mathcal{O}_{X \times X}}^2(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}).$$

The map (0.5) has a left inverse  $u : Ext_{\mathcal{O}_{X \times X}}^2(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}) \rightarrow H^2(X, \mathcal{O}_X)$  which takes  $\beta$  to  $\beta_{\mathcal{O}_X}$ . But, by the Lemma, the later class is equal to 0 since  $\mathcal{O}_X$  is liftable modulo  $p^2$ . It follows that  $\beta$  is 0.

On the other hand, let  $\Gamma \subset X = \overline{Z} \times \overline{Z}$  be the graph of the Frobenius morphism  $Fr : \overline{Z} \rightarrow \overline{Z}$  and  $\mathcal{O}_\Gamma$  the structure sheaf of  $\Gamma$  viewed as a coherent sheaf on  $X$ . Then, by our first assumption, the sheaf  $\mathcal{O}_\Gamma$  is not liftable modulo  $p^2$ . Hence, by the Lemma,  $\beta_{\mathcal{O}_\Gamma}$  is not 0. This contradiction completes the proof.  $\square$

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